On the minimum polynomial of supermatrices

2002 J. Phys. A: Math. Gen. 35 9183

(http://iopscience.iop.org/0305-4470/35/43/313)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) 9183–9197 PII: S0305-4470(02)36287-5

On the minimum polynomial of supermatrices

Anargyros G Fellouris and Lina K Matiadou

Department of Mathematics and Physical Sciences, National Technical University of Athens, Zografou Campus, 157 80 Athens, Greece

E-mail: afellou@math.ntua.gr

Received 26 April 2002, in final form 29 July 2002
Published 15 October 2002
Online at stacks.iop.org/JPhysA/35/9183

Abstract
In this paper, a new selection of factors for the construction of the minimum polynomial of a supermatrix M is proposed, leading to null polynomials of M of lower degree than the degree of the corresponding polynomial obtained by using the method proposed in the work of Urrutia and Morales [1]. The case of $(1 + 1) \times (1 + 1)$ supermatrices has been completely discussed. Moreover, the main theorem concerning the construction of the minimum polynomial as a product of factors from the characteristic polynomial in the general case of $(m+n) \times (m+n)$ supermatrices is given. Finally, we prove that the minimum polynomial of a supermatrix M, in general, is not unique.

PACS numbers: 02.10.Yn, 02.10.−v, 02.20.−a

1. Introduction

For any matrix M over the field F, $F = \mathcal{R}$ or $\mathcal{C}$, its characteristic polynomial is defined by $\mathcal{X}_M(t) = \det(tI - M)$, where I is the $n \times n$ identity matrix. According to the *Cayley–Hamilton theorem*, every matrix M satisfies its characteristic polynomial, that is, if we substitute the indeterminate t by the matrix M using $t^0 = I$, the produced polynomial matrix $\mathcal{X}_M(M)$ is the zero matrix. The characteristic polynomial can be written in the form

$$\mathcal{X}_M(t) = t^n + c_{n-1}t^{n-1} + \cdots + c_1 t + c_0 \tag{1.1}$$

with $c_0, c_1, \ldots, c_{n-1} \in F$. The coefficients $c_0, c_1, \ldots, c_{n-1}$ are invariants of the matrix M under similarity and they can be written in terms of traces of M and its powers $M^2, \ldots, M^{n-1}$. Especially, we have

$$c_0 = (-1)^n \det M \qquad c_{n-1} = -\operatorname{tr} M. \tag{1.2}$$

The *Cayley–Hamilton theorem* has found many interesting applications as follows:

- in the construction of the so-called skein relations, which are relevant to the calculation of expectation values, and in the process of reduction of the phase space [2];

- in the (2 + 1)-dimensional *Chern–Simons theories* [3];
- in the discussion of the reduced phase space of the *de Sitter gravity* in (2 + 1) dimensions, which is equivalent to the *Chern–Simons theory* of the group $SO(2, 2)$ [4];
- in the reduction of phase space in the case of (2 + 1) *super de Sitter gravity*, which is the *Chern–Simons theory* of the supergroup $Osp(1/2, \mathcal{C})$ [5];
- in the study of loop representations in quantum supergravity in terms of a GSU(2) connection, in order to state that any product of Wilson loops can be expressed as a linear combination of Wilson loops [6].

This paper is motivated from the work of Urrutia and Morales [1], where the Cayley–Hamilton theorem for supermatrices is discussed and a method for the construction of the minimum polynomial of a supermatrix is introduced. In section 2, we introduce our notation concerning all well-known material on *Grassmann* algebras, supermatrices and their characteristic polynomial.

In section 3, we study completely the case of $(1 + 1) \times (1 + 1)$ supermatrices. We determine the minimum polynomial of the arbitrary $(1+1) \times (1+1)$ supermatrix (theorem 3.2) and prove that it is, in general, not unique.

In section 4, we include our proposal on the construction of null polynomials for a supermatrix M, which gives the possibility of constructing a null polynomial of a supermatrix M of degree less than the degree of the corresponding minimum polynomial introduced by the method proposed by Urrutia and Morales [1].

We prove that the minimum polynomial of a supermatrix is a divisor of its characteristic polynomial and state the general theorem. We use three examples to describe the general situation on the minimum polynomial of a supermatrix.

2. Null polynomials for supermatrices

Let Λ_p denote the Grassmann algebra on $p < +\infty$ mutually anticommuting generators, over the field F of scalars (i.e. the real or complex numbers). Λ_p can be written as the direct sum of two subspaces $\Lambda_{p,\bar{0}} \oplus \Lambda_{p,\bar{1}}$, where $\Lambda_{p,\bar{0}}$ (resp. $\Lambda_{p,\bar{1}}$) is the even (resp. odd) part of Λ_p and consists of all linear combinations of products of an even (resp. odd) number of generators. $\Lambda_{p,\bar{0}}$ contains the identity $\mathbf{1}$ and its elements commute with the elements of Λ_p. The elements of $\Lambda_{p,\bar{1}}$ mutually anticommute.

Alternatively, Λ_p can be written as the direct sum $\Lambda_p = F \bigoplus \Lambda_{p,N}$ where F is the field of scalars and $\Lambda_{p,N}$ consist of all linear combinations of a non-zero number of generators. Thus, any element $\alpha \in \Lambda$ is a sum of the form $\alpha = \bar{\alpha} + s(\alpha)$ where $\bar{\alpha} \in F$ is the *body* or *numeric part* of α and $s(\alpha)$ is the *soul* or nilpotent part of α. The elements of $\Lambda_{p,N}$ are nilpotent with degree of nilpotency less than or equal to p. The invertible elements of Λ_p are of the form $f\mathbf{1} + n$, where $f \neq 0$ and $n \in \Lambda_{p,N}$ and constitute a subgroup Λ_p^* of Λ_p. For further details, see [7].

Rogers in [8] has shown that Λ_p has a norm which gives it the structure of a *Banach algebra*. Rogers further has shown that p can be taken to be infinity and in that case Λ_∞ consists of all those linear combinations of products, with a finite number of factors in each product, from a countable set of anticommuting generators, which have a finite norm. Λ_∞ is a Banach algebra which retains some but not all of the algebraic properties of Λ_p for $p < \infty$. For example, the elements in $\Lambda_{\infty,\bar{0}}$ commute with all elements of Λ_∞, the elements of $\Lambda_{\infty,\bar{1}}$ mutually anticommute and they are nilpotent of degree 2. In general, the elements of $\Lambda_{\infty,N}$ are not nilpotent in the algebraic sense, but they are topologically nilpotent, see [8]. In the following we use Λ, instead of Λ_p, $p \leqslant \infty$.

Let

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{2.1}$$

be a supermatrix over Λ, of type $(m+n) \times (m+n)$, i.e. A (resp. D) is an $m \times m$ (resp. $n \times n$) matrix whose entries belong to $\Lambda_{\bar{0}}$ and B (resp. C) is an $m \times n$ (resp. $n \times m$) matrix whose entries belong to $\Lambda_{\bar{1}}$. The set of all supermatrices of the form (2.1) is denoted by $\mathcal{M}(m, n;\, \Lambda)$ and it is the even part of the *Grassmannification* of the *Lie superalgebra* $\mathcal{M}(m, n;\, F)$ given by $\Lambda \otimes \mathcal{M}(m, n;\, F)$, i.e.

$$\begin{aligned} \mathcal{M}(m, n;\, \Lambda) &= (\Lambda \otimes \mathcal{M}(m, n;\, F))_{\bar{0}} \\ &= (\Lambda_0 \otimes \mathcal{M}(m, n;\, F)_{\bar{0}}) \oplus (\Lambda_1 \otimes \mathcal{M}(m, n;\, F)_{\bar{1}}) \end{aligned}$$

a form which explicitly displays the decomposition into diagonal and off-diagonal components. It is called the *Grassmann envelope* [9] of the Lie superalgebra $\mathcal{M}(m, n;\, F)$.

It is known from the work of Nieuwenhuizen [10] that the supermatrix M is invertible if and only if A and D are invertible, which is the case if and only if the matrices $\bar{A}$ and $\bar{D}$ formed from the numeric components (bodies) of the entries in A and D are invertible. Explicitly, we have the following two equivalent forms [11]:

$$\begin{aligned} M^{-1} &= \begin{bmatrix} A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{bmatrix} \\ &= \begin{bmatrix} (A - BD^{-1}C)^{-1} & -(A - BD^{-1}C)^{-1}BD^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & D^{-1}C(A - BD^{-1}C)^{-1}BD^{-1} + D^{-1} \end{bmatrix}. \end{aligned} \tag{2.2}$$

There is a linear $\Lambda_{\bar{0}}$-valued function on the set of all supermatrices $\mathcal{M}(m, n;\, \Lambda)$, called *supertrace* and defined by

$$\operatorname{str} M = \operatorname{tr} A - \operatorname{tr} D \tag{2.3}$$

where the trace function is the usual sum of the diagonal elements of a square matrix (see Arnowitt *et al* [12]).

The *superdeterminant* or *Berezinian* of a supermatrix M is a $\Lambda_{\bar{0}}$-valued function defined by

$$s \det M = (\det A) \det{}^{-1}(D - CA^{-1}B) \tag{2.4}$$

for invertible supermatrices. An equivalent formula of the superdeterminant is [13]

$$s \det M = \det(A - BD^{-1}C) \det{}^{-1} D. \tag{2.5}$$

In the following, we use polynomials from the ring of polynomials $\Lambda_{\bar{0}}[x]$, which is not an integral domain [14]. We denote by $\mathcal{R} = s(\Lambda_{\bar{0}})[x]$ the set of nilpotent elements of $\Lambda_{\bar{0}}[x]$. It is the smallest prime ideal of $\Lambda_{\bar{0}}[x]$. The ratio of polynomials is an element of the quotient ring, the localization of $\Lambda_{\bar{0}}[x]$ at the minimal prime ideal. It is defined by

$$\Lambda_{\bar{0}}(x) = \Lambda_{\bar{0}}[x]_{\mathcal{R}} = \left\{ \frac{f}{g} : f, g \in \Lambda_{\bar{0}}[x],\, g \notin \mathcal{R} \right\}. \tag{2.6}$$

$\Lambda_{\bar{0}}(x)$ is the even part of the $\mathcal{Z}_2$-graded algebra

$$\Lambda(x) = \left\{ \frac{f}{g} : f \in \Lambda[x],\, g \in \Lambda_{\bar{0}}[x],\, g \notin \mathcal{R} \right\} \tag{2.7}$$

where $\Lambda[x]$ is the ring of polynomials over the Grassmann algebra Λ.

The body $\bar{f}$ of a polynomial

$$f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_0 \tag{2.8}$$

in $\Lambda[x]$ or in $\Lambda_{\bar{0}}[x]$ is defined by

$$\bar{f}(x) = \bar{a}_0 x^n + \bar{a}_1 x^{n-1} + \cdots + \bar{a}_{n-1} x + \bar{a}_0. \tag{2.9}$$

A polynomial $f(x) \in \Lambda[x]$ is invertible, if $\bar{f} \neq 0$.

For a rational function $h(x) = \frac{f(x)}{g(x)} \in \Lambda_{\bar{0}}(x)$ and $w \in \Lambda_{\bar{0}}$ we have

$$h(w) = \begin{cases} \dfrac{f(w)}{g(w)} & \text{if } \quad \overline{g(w)} \neq 0 \\ \text{it is not defined} & \text{otherwise.} \end{cases} \tag{2.10}$$

$w \in \Lambda_0$ is a zero of h, if $h(w) = 0$, $w \in \Lambda_0$ is a pole of h, if h is invertible and $h^{-1}(w) = 0$.

Any supermatrix of the form (2.1) can be written as

$$M = \bar{M} + s(M) \tag{2.11}$$

where

$$\bar{M} = \begin{bmatrix} \bar{A} & 0 \\ 0 & \bar{D} \end{bmatrix} \qquad s(M) = \begin{bmatrix} s(A) & B \\ C & s(D) \end{bmatrix} \tag{2.12}$$

with $\bar{A} = (\bar{a}_{ij})$, $\bar{D} = (\bar{d}_{ij})$, $s(A) = (s(a_{ij}))$, $s(D) = (s(d_{ij}))$.

An even vector is a column

$$X_0 = (x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+n})^T \tag{2.13}$$

where $x_i \in \Lambda_{\bar{0}}$, for $i = 1, 2, \ldots, m$, and $x_i \in \Lambda_{\bar{1}}$, for $i = m+1, \ldots, m+n$.

An odd vector is a column

$$X_1 = (x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+n})^T \tag{2.14}$$

where $x_i \in \Lambda_{\bar{1}}$, for $i = 1, 2, \ldots, m$, and $x_i \in \Lambda_{\bar{0}}$, for $i = m+1, \ldots, m+n$. The number $\lambda \in \Lambda_{\bar{0}}$ is an eigenvalue of the supermatrix M [15], if there exists a vector X of the form (2.13) or (2.14) with $\bar{X} \neq 0$ such that

$$MX = \lambda X. \tag{2.15}$$

The eigenvalue λ is of the first (resp. second) kind, if the corresponding eigenvector X is even (resp. odd) [15].

The supermatrix M of the form (2.1) is called separable or generic [9], if the eigenvalues $\alpha_1, \alpha_2, \ldots, \alpha_m$ of $\bar{A}$ and the eigenvalues $\delta_1, \delta_2, \ldots, \delta_n$ of $\bar{D}$ are all different. Then M has eigenvalues $a_1, a_2, \ldots, a_m$ of the first kind such that $\bar{a}_1 = \alpha_1, \bar{a}_2 = \alpha_2, \ldots, \bar{a}_m = \alpha_m$, and eigenvalues $d_1, d_2, \ldots, d_n$ of the second kind such that $\bar{d}_1 = \delta_1, \bar{d}_2 = \delta_2, \ldots, \bar{d}_n = \delta_n$. Moreover, there exists an invertible supermatrix N such that

$$NMN^{-1} = \mathrm{diag}(a_1, a_2, \ldots, a_m, d_1, d_2, \ldots, d_n). \tag{2.16}$$

The characteristic function of a supermatrix M given by (2.1) is a rational function of an even Grassmann indeterminant x defined by Kobayashi and Nagamachi [14], as follows

$$h(x) = s\det(xI - M) = \frac{F(x)}{G(x)} = \frac{\tilde{F}(x)}{\tilde{G}(x)} \tag{2.17}$$

where we have

$$\begin{aligned} F(x) &= \det(xI - A)^{n+1} \\ G(x) &= \det[\det(xI - A)(xI - D) - C(\det(xI - A)(xI - A)^{-1}B] \\ \tilde{F}(x) &= \det[\det(xI - D)(xI - A) - B(\det(xI - D)(xI - D)^{-1}C] \\ \tilde{G}(x) &= \det(xI - D)^{m+1}. \end{aligned}$$

Then, the *characteristic polynomial* $P(x)$ of the supermatrix M is defined as [1]

$$P(x) = \tilde{F}(x)G(x) = F(x)\tilde{G}(x) = a(x)^{n+1}d(x)^{m+1} \tag{2.18}$$

where we have put

$$a(x) = \det(xI - A) \tag{2.19}$$

$$d(x) = \det(xI - D). \tag{2.20}$$

The well-known Cayley–Hamilton theorem for a usual real or complex matrix is also valid for supermatrices with the characteristic polynomial $P(x)$ defined by (2.18) [16].

Urrutia and Morales [1] in their attempt to determine for the supermatrix M, given by (2.1), its *minimum* polynomial, that is, a monic null polynomial $Q(x)$ of the least possible degree, such that $Q(M) = 0$, define the polynomial

$$Q(x) = \tilde{f}(x)g(x) = f(x)\tilde{g}(x) \tag{2.21}$$

where $f, \tilde{f}, g, \tilde{g}$ are coming from the relations

$$\tilde{F} = R\tilde{f} \qquad \tilde{G} = R\tilde{g} \qquad F = Sf \qquad G = Sg \tag{2.22}$$

where R (resp. S) is a common divisor of maximum degree of the pair $\tilde{F}, \tilde{G}$ (resp. F, G).

The polynomial $Q(x)$ is a null polynomial of the supermatrix M, for any common factors R and S satisfying (2.22) ([1], theorem 3.2).

Our unease, for the consideration of the common factors R, S, is that their elimination in the definition of the polynomial $Q(x)$, even it leads to a null polynomial of the supermatrix M of lower degree than the characteristic polynomial $P(x)$, has the disadvantage that it eliminates some factors involved in the computation of the superdeterminant, as well as in the definition of the characteristic polynomial $P(x)$. It looks possible sometimes, using factors from the characteristic polynomial $P(x)$, the factors of R, S included, to construct a null monic polynomial of the supermatrix M of lower degree than the degree of the polynomial $Q(x)$.

Next, we prove that the assertion of Urrutia and Morales ([1], theorem 3.2), that the polynomial $Q(x)$ given by (2.21) is a null polynomial of the supermatrix M does not generally work, even if we consider the simplest case of $(1 + 1) \times (1 + 1)$ supermatrices.

3. The case of (1 + 1) × (1 + 1) supermatrices

We consider the arbitrary $(1 + 1) \times (1 + 1)$ supermatrix

$$M = \begin{bmatrix} p & \alpha \\ \beta & q \end{bmatrix} \qquad p, q \in \Lambda_{\bar{0}} \quad \alpha, \beta \in \Lambda_{\bar{1}}.$$

Then we have

$$\tilde{F}(x) = (x - p)(x - q) - \alpha\beta \qquad \tilde{G}(x) = (x - q)^2$$
$$F(x) = (x - p)^2 \qquad G(x) = (x - p)(x - q) + \alpha\beta.$$

Bearing in mind that $\Lambda_{\bar{0}}[x]$ is not a unique factorization ring and following factorizations and the Euclidean algorithm [13], we consider the cases:

Case 1 ($p \neq q$). We now distinguish the cases:

Case 1a ($\bar{p} \neq \bar{q}, \alpha\beta \neq 0$). Then following the factorization theory developed in [13] we have

$$\tilde{F}(x) = (x - p)(x - q) - \alpha\beta = \left(x - p + \frac{\alpha\beta}{q - p}\right)\left(x - q - \frac{\alpha\beta}{q - p}\right)$$

$$\tilde{G}(x) = (x - q)^2 = \left(x - q + \frac{\alpha\beta}{q - p}\right)\left(x - q - \frac{\alpha\beta}{q - p}\right)$$

$$F(x) = (x-p)^2 = \left(x - p + \frac{\alpha\beta}{q-p}\right)\left(x - p - \frac{\alpha\beta}{q-p}\right)$$
$$G(x) = (x-p)(x-q) + \alpha\beta = \left(x - p - \frac{\alpha\beta}{q-p}\right)\left(x - q + \frac{\alpha\beta}{q-p}\right).$$

We note that the factorization of the polynomials $\tilde{F}(x)$, $G(x)$ is unique, while the factorization of $\tilde{G}(x)$, $F(x)$, which is not unique, has been chosen in the given form in order to obtain the common factors:

$$R = x - q - \frac{\alpha\beta}{q-p} \qquad S = x - p - \frac{\alpha\beta}{q-p}.$$

The characteristic function $h(x)$ is simplified to the form

$$h(x) = \frac{\tilde{f}}{\tilde{g}} = \frac{f}{g} = \frac{x - p + \frac{\alpha\beta}{q-p}}{x - q + \frac{\alpha\beta}{q-p}}.$$

Thus, we have the characteristic polynomial

$$P(x) = \tilde{F}(x)G(x) = F(x)\tilde{G}(x) = (x-p)^2(x-q)^2$$

and the polynomial

$$Q(x) = \tilde{f}(x)g(x) = f(x)\tilde{g}(x) = \left(x - p + \frac{\alpha\beta}{q-p}\right)\left(x - q + \frac{\alpha\beta}{q-p}\right)$$

which is a monic null polynomial of the supermatrix M. It can easily be checked that the polynomial $Q(x)$ is the minimum polynomial of the supermatrix M and that it is unique.

Case 1b ($\bar{p} \neq \bar{q}, \alpha\beta = 0$). Then we have

$$P(x) = \tilde{F}(x)G(x) = F(x)G(x) = (x-p)^2(x-q)^2$$
$$R(x) = x - p \qquad S(x) = x - q$$

and the polynomial of degree two

$$Q(x) = \tilde{f}(x)g(x) = f(x)\tilde{g}(x) = (x-p)(x-q)$$

is, in fact, the minimum polynomial of the supermatrix M.

Case 1c ($\bar{p} = \bar{q}, \alpha\beta \neq 0$). Then we have

$$P(x) = \tilde{F}(x)G(x) = F(x)\tilde{G}(x) = (x-p)^2(x-q)^2 \qquad R(x) = 1 \qquad S(x) = 1.$$

The characteristic function cannot be simplified and the degree-four polynomial

$$Q(x) = (x-p)^2(x-q)^2 = P(x)$$

is the minimum polynomial of the supermatrix M.

Case 2 ($p = q$). We distinguish the cases:

Case 2a ($\alpha\beta \neq 0$). Then we have the characteristic function

$$h(x) = \frac{\tilde{F}(x)}{\tilde{G}(x)} = \frac{(x-p)^2 - \alpha\beta}{(x-p)^2} = \frac{(x-p)^2}{(x-p)^2 + \alpha\beta} = \frac{F(x)}{G(x)}$$

and the characteristic polynomial

$$P(x) = (x-p)^4 = Q(x)$$

because the characteristic function cannot be simplified. Clearly, $Q(x)$ is a null polynomial of the supermatrix M. However, we can check that the minimum polynomial of M is the third degree polynomial

$$m(x) = (x - p)^3.$$

Moreover, we can check that $m(x)$ is not unique, as we can easily verify that the minimum polynomial $m_{\bar{M}}(x) = x - \bar{p}$ of the numeric part $\bar{M}$ of the supermatrix M, multiply by a factor $\sigma \in \Lambda_{\bar{0}}$ annihilating all nilpotent elements in $\Lambda_{\bar{0}}$, is also a null polynomial of M. Therefore, we have a family of monic null polynomials of the supermatrix M of degree three

$$m(x) + \kappa\sigma m_{\bar{M}}(x) = (x - p)^3 + \kappa\sigma(x - \bar{p}) \qquad \kappa \in \mathcal{C}.$$

Later we will see that it is valid generally (theorem 3.1).

Case 2b ($\alpha\beta = 0$). Then we have the characteristic function

$$h(x) = \frac{(x - p)^2}{(x - p)^2} = 1$$

and the characteristic polynomial

$$P(x) = (x - p)^4.$$

In this case, we have $R(x) = (x - p)^2 = S(x)$ and thus $Q(x) = 1$, which is not a null polynomial of the supermatrix M. With an easy calculation we find that the polynomial

$$m(x) = (x - p)^2$$

of degree two is the minimum polynomial of the supermatrix M. According to the discussion of non-uniqueness made in case 2a, also in this case the minimum polynomial $m(x)$ is not unique.

Case 2c ($\alpha = 0 = \beta$). In this case, we have

$$h(x) = 1 \qquad R(x) = S(x) = (x - p)^2 \qquad P(x) = (x - p)^4$$

and $Q(x) = 1$, which is not a null polynomial of the supermatrix M. The minimum polynomial of M is the first degree polynomial

$$m(x) = x - p$$

which is unique.

From the previous discussion it is clear that the use of the common factors $R(x)$ and $S(x)$ in the case of a non-separable $(1 + 1) \times (1 + 1)$ supermatrix M, either does not lead to a null polynomial of the supermatrix M or does not lead to the minimum polynomial of M.

In all cases the minimum polynomial of M is a product of factors of the characteristic polynomial $P(x)$ of the form

$$a(x) = x - p \qquad d(x) = x - q \qquad a(x) \pm \alpha\beta w_1(x) \qquad \text{and} \qquad d(x) \pm \alpha\beta w_2(x)$$

where $w_i(x) \in \Lambda_0[x]$, with $\deg w_i(x) \leqslant 1$, $i = 1, 2$, which are taken from the various possible factorizations of $P(x)$. Factors of the form

$$a(x) \pm \alpha\beta w_1(x) \qquad \text{and} \qquad d(x) \pm \alpha\beta w_2(x)$$

with $w_i(x) \in \Lambda_0[x]$, $\deg w_i(x) \geqslant 2$ are not suitable because these do not lead to monic polynomials.

In the case of separable $(1 + 1) \times (1 + 1)$ supermatrix M, i.e. when $\bar{p} \neq \bar{q}$, the minimum polynomial of M is the second degree polynomial

$$m(x) = \left(x - p + \frac{\alpha\beta}{q - p}\right)\left(x - q + \frac{\alpha\beta}{q - p}\right)$$

where $p - \frac{\alpha\beta}{q-p}$ is the eigenvalue of the first kind and $q - \frac{\alpha\beta}{q-p}$ is the eigenvalue of the second kind of the supermatrix M. In this case, the method of simplification of the characteristic function, proposed in [1], is effective. However, this method is not effective in the case of an $(1 + 1) \times (1 + 1)$ non-separable supermatrix M. In that case, the minimum polynomial $m(x)$ of M can be of every possible degree, that is

$$1 \leqslant \deg m(x) \leqslant 4 = \deg P(x).$$

Furthermore, in the case of $(1 + 1) \times (1 + 1)$ non-separable supermatrix, the minimum polynomial is in general not unique, according to the following general result.

Theorem 3.1. *For every supermatrix M over $\Lambda = \Lambda_p$, $p \leqslant +\infty$, if $M = \bar{M} + M_N$ and $m_{M_0}(x)$ is the minimum polynomial of the numeric part M_0, then*

$$\sigma m_{\bar{M}}(M) = 0$$

where $\sigma \in \Lambda_{\bar{0}}$ is an element annihilating all non-invertible elements in Λ.

Proof. Let $M = \bar{M} + M_N$ be an $(m+n) \times (m+n)$ supermatrix, when $\bar{M}$ is an $(m+n) \times (m+n)$-matrix over $\mathcal{C}$, called the numeric part of M. Let

$$m(t) = t^k + a_{k-1}t^{k-1} + \cdots + a_1 t + a_0$$

where $a_0, a_1, \ldots, a_{k-1} \in \mathcal{C}$, $k \leqslant m + n$, be the minimum polynomial of $\bar{M}$. We consider the polynomial

$$m(M) = M^k + a_{k-1}M^{k-1} + \cdots + a_1 M + a_0 I.$$

We observe that

$$M^l = (\bar{M} + M_N)^l = \bar{M}^l + M_{(N,l)} \qquad l = 1, 2, \ldots, k$$

where $M_{(N,l)}$ is an $(m + n) \times (m + n)$-supermatrix having all its elements nilpotent, as it is a finite sum of supermatrices which are products of supermatrices with the term M_N appearing at least once. Therefore, we have

$$\sigma M_{(N,l)} = 0 \tag{3.1}$$

for every $l = 1, 2, \ldots, k$, where σ is an even Grassmann number annihilating all nilpotent elements in Λ.

Finally, we write

$$\begin{aligned} m(M) &= (\bar{M}^k + a_{k-1}\bar{M}^{k-1} + \cdots + a_0 I) + M_{(N,k)} + a_{k-1}M_{(N,k-1)} + \cdots + a_1 M_{(N,1)} \\ &= m(\bar{M}) + M_{(N,k)} + a_{k-1}M_{(N,k-1)} + \cdots + a_1 M_{(N,1)} \\ &= M_{(N,k)} + a_{k-1}M_{(N,k-1)} + \cdots + a_1 M_{(N,1)} \end{aligned}$$

and therefore by (3.1) it is obvious that

$$\sigma m(M) = 0.$$

□

From all the above, we have proved the following theorem concerning the case of $(1 + 1) \times (1 + 1)$-supermatrices.

Theorem 3.2. *Let*

$$M = \begin{bmatrix} p & \alpha \\ \beta & q \end{bmatrix} \qquad p, q \in \Lambda_{\bar{0}} \quad \alpha, \beta \in \Lambda_{\bar{1}}$$

be the arbitrary $(1 + 1) \times (1 + 1)$-supermatrix with the characteristic function

$$h(x) = \frac{(x - p)(x - q) - \alpha\beta}{(x - q)^2} = \frac{(x - p)^2}{(x - p)(x - q) + \alpha\beta}$$

and the characteristic polynomial

$$P(x) = (x - p)^2(x - q)^2.$$

Then one has the following cases:

- *If M is separable, then the minimum polynomial of M is of the form*

$$m(x) = \left(x - p + \frac{\alpha\beta}{q-p}\right)\left(x - q + \frac{\alpha\beta}{q-p}\right)$$

where $p - \frac{\alpha\beta}{q-p}$ *and* $q - \frac{\alpha\beta}{q-p}$ *are the eigenvalues of M of the first and second kinds, respectively.*

- *If M is not separable, i.e.* $\bar{p} = \bar{q}$*, but* $p \neq q$*, then the minimum polynomial of M is of the form*

$$m(x) = (x - p)^2(x - q)^2.$$

- *If M is not separable and* $p = q$*, then the minimum polynomial of M is of the form*

$$m(x) = \begin{cases} (x-p)^3 & \text{when} \quad \alpha\beta \neq 0 \\ (x-p)^2 & \text{when} \quad \alpha\beta = 0 \\ (x-p) & \text{when} \quad \alpha = \beta = 0. \end{cases}$$

- *If M is non-separable its minimum polynomial is in general not unique.*

4. The general case

We consider the arbitrary $(m + n) \times (m + n)$-supermatrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

with characteristic function given by (2.17) as

$$h(x) = \frac{\tilde{F}(x)}{\tilde{G}(x)} = \frac{F(x)}{G(x)}$$

and characteristic polynomial given by (2.18) as

$$P(x) = \tilde{F}(x)G(x) = F(x)\tilde{G}(x).$$

Suppose that $m(x)$ is the minimum polynomial of M, that is, a monic null polynomial of M of the least possible degree. Then $\deg m(x) \leqslant \deg P(x)$. According to the Euclidean algorithm [17], applied to $P(x)$ and $m(x)$ in the ring of polynomials $\Lambda_{\bar{0}}[x]$, there exist unique polynomials $\Pi(x)$ and $v(x)$ such that

$$P(x) = m(x)\Pi(x) + v(x) \tag{4.1}$$

and $\deg v(x) < \deg m(x)$ or $v(x) = 0$. Since $P(x)$ and $m(x)$ are null polynomials of M, if $v(x) \neq 0$, then $v(x)$ will be a null polynomial of M of degree less than the degree of $m(x)$, which is absurd. Hence $v(x) = 0$ and therefore $m(x)$ divides $P(x)$.

We note that $\Lambda_{\bar{0}}[x]$ is not a unique factorization ring. In particular, factors $u(x)^{2r}$ of even degree can be written as

$$u(x)^{2r} = [u(x)^r + \theta v(x)][u(x)^r - \theta v(x)] \tag{4.2}$$

where $\theta \in \Lambda_{\bar{0}}$ with $\theta^2 = 0$ and $v(x)$ is arbitrary in $\Lambda_{\bar{0}}[x]$. From the previous discussion we have that $m(x)$ must be a product of factors taken from a factorization of the characteristic polynomial $P(x)$. Some factors can be of the form given in the right-hand side of (4.2).

Therefore from the above discussion, theorem 3.1 and the work of Urrutia and Morales [1] and [16], we have proved the general theorem for the minimum polynomial of the arbitrary $(m+n)\times(m+n)$ supermatrix M. It can be stated as follows.

Theorem 4.1. *Let*

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

be the arbitrary $(m+n)\times(m+n)$*-supermatrix with the characteristic polynomial*

$$P(x) = \tilde{F}(x)G(x) = a(x)^{n+1}d(x)^{m+1}$$

where

$$a(x) = \det(xI - A) \qquad d(x) = \det(xI - D).$$

Let

$$a_1(x), a_2(x), \ldots, a_r(x) \qquad \textit{and} \qquad d_1(x), d_2(x), \ldots, d_s(x)$$

be the irreducible factors of the polynomials $a(x), d(x)$ *so that*

$$a(x) = a_1(x)^{i_1}a_2(x)^{i_2}\ldots a_r(x)^{i_r}$$
$$d(x) = d_1(x)^{j_1}d_2(x)^{j_2}\ldots d_s(x)^{j_s}.$$

Then $P(x)$ *is a null polynomial of* M. *The minimum polynomial* $m(x)$ *of* M *divides* $P(x)$ *and is of the form*

$$m(x) = m_1(x)m_2(x)\ldots m_k(x)$$

where $m_i(x), i = 1, 2, \ldots, k$ *is one of the following factors:*

$$a_\mu(x)^{k_\mu}, \qquad d_\nu(x)^{l_\nu}, \qquad 0 \leqslant k_\mu \leqslant i_\mu(n+1), \qquad 0 \leqslant l_\nu \leqslant j_\nu(m+1)$$
$$A(x)D(x) \pm \theta\omega(x)$$

where

$$\theta \in \Lambda_{\bar{0}} \qquad \textit{with} \quad \theta^2 = 0$$
$$\omega(x) \in \Lambda_{\bar{0}}[x] \qquad \textit{with} \quad \deg\omega(x) \leqslant \deg A(x)D(x)$$
$$A(x) = \prod_{\mu=1}^{r} a_\mu(x)^{k_\mu} \qquad D(x) = \prod_{\nu=1}^{s} d_\nu(x)^{l_\nu}$$
$$0 \leqslant k_\mu \leqslant i_\mu(n+1) \qquad 0 \leqslant l_\nu \leqslant j_\nu(m+1).$$

In the following, we provide some examples to explain the arbitrariness of the selection of factors for the construction of the minimum polynomial of a supermatrix M for any possible factorization of the characteristic polynomial.

Example 1. For the supermatrix [1],

$$M = \begin{pmatrix} 0 & 0 & 0 & \theta_1 \\ 0 & 1 & \theta_2 & 0 \\ 0 & \theta_1 & -1 & 0 \\ \theta_2 & 0 & 0 & 0 \end{pmatrix} \tag{4.3}$$

with $\theta_1, \theta_2 \in \Lambda_{\bar{1}}$, we have

$$\tilde{F}(x) = x^3(x+1)^2(x-1) + \theta x(x+1) \qquad \text{where} \quad \theta = \theta_1\theta_2$$
$$\tilde{G}(x) = x^3(x+1)^3$$
$$F(x) = x^3(x-1)^3$$
$$G(x) = x^3(x-1)^2(x+1) - \theta x(x-1).$$

Using the common factors R, S, the minimum polynomial is determined as

$$Q(x) = x^6 + \theta x^5 - x^4.$$

However, using an alternative approach based on all factors involved in the definition of the characteristic polynomial, one is able to find a null monic polynomial of the supermatrix M of degree lower than 6. More explicitly, we have

$$a(x) = x(x-1)$$
$$d(x) = x(x+1)$$
$$\tilde{F}(x) = x(x+1)[x^2(x^2-1)+\theta]$$
$$G(x) = x(x-1)[x^2(x^2-1)-\theta].$$

Thus the characteristic polynomial is

$$P(x) = \tilde{F}(x)G(x) = x^2(x-1)(x+1)[x^2(x^2-1)+\theta][x^2(x^2-1)-\theta]$$
$$= x^6(x-1)^3(x+1)^3.$$

We observe that

$$M(M+I)[M^2(M^2-I)+\theta I] = 0 \tag{4.4}$$
$$M(M-I)[M^2(M^2-I)-\theta I] = 0. \tag{4.5}$$

Thus, we find for the supermatrix M, the null polynomials

$$P_1(x) = x(x+1)[x^2(x^2-1)+\theta]$$
$$P_2(x) = x(x-1)[x^2(x^2-1)-\theta]$$

as well as the null polynomials

$$Q_1(x) = \tfrac{1}{2}(P_1(x)+P_2(x)) = x^6 - x^4 + \theta x$$
$$Q_2(x) = \tfrac{1}{2}(P_1(x)-P_2(x)) = x^5 - x^3 + \theta x^2.$$

Hence, we have determined a null monic polynomial $Q_2(x)$ of the supermatrix M such that

$$\text{degree } Q_2(x) < \text{degree } Q(x).$$

Moreover, we have that

$$Q_2(x) = x^2[x(x+1)(x-1)+\theta]$$

which is a product of factors of the characteristic polynomial $P(x)$, because of the factorization

$$P(x) = x^2(x^2-1)^2x^4(x^2-1)$$
$$= [x(x^2-1)+\theta][x(x^2-1)-\theta]x^4(x^2-1).$$

Finally, we check that an equality of the form

$$M^4 + k_3M^3 + k_2M^2 + k_1M + k_0I = 0$$

with k_0, k_1, k_2, k_3 in $\Lambda_{\bar{0}}$ is impossible. Hence, the polynomial $Q_2(x)$ is a null polynomial of M of the least possible degree.

Moreover, the minimum polynomial of the numeric part $\bar{M}$ of M is given by

$$m_{\bar{M}}(x) = x(x+1)(x-1).$$

Hence, according to the theorem 3.1 every polynomial of the form

$$Q_2(x) + \lambda\theta m_{\bar{M}}(x) \qquad \lambda \in \mathcal{C}$$

is a monic null polynomial of M of minimum degree.

Example 2. For the supermatrix

$$M = \begin{pmatrix} \theta & 0 & 0 & \theta_1 \\ 0 & \theta & \theta_2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.6}$$

where $\theta_1, \theta_2 \in \Lambda_1$ and $\theta = \theta_1\theta_2$, we have

$$\begin{aligned} a(x) &= (x-\theta)^2 & d(x) &= x(x-1) \\ \tilde{F}(x) &= x^2(x-1)^2(x-\theta)^2 & \tilde{G}(x) &= x^3(x-1)^3 \\ F(x) &= (x-\theta)^6 & G(x) &= x(x-1)(x-\theta)^4. \end{aligned}$$

Thus we obtain

$$\frac{\tilde{F}}{\tilde{G}} = \frac{x^2(x-1)^2(x-\theta)^2}{x^3(x-1)^3} = \frac{(x-\theta)^2}{x(x-1)} = \frac{x^2-2\theta x}{x(x-1)} = \frac{x-2\theta}{x-1} = \frac{\tilde{f}}{\tilde{g}}$$

$$\frac{F}{G} = \frac{(x-\theta)^6}{x(x-1)(x-\theta)^4} = \frac{(x-\theta)^2}{x(x-1)} = \frac{x-2\theta}{x-1} = \frac{f}{g}.$$

The polynomial that is obtained according to the method proposed in [1], is the polynomial

$$Q(x) = \tilde{f}g = f\tilde{g} = (x-1)(x-2\theta)$$

which unfortunately is not a null polynomial of the supermatrix M. However, using factors from the polynomial

$$P(x) = \tilde{F}(x)G(x) = x^3(x-1)^3(x-\theta)^6$$

we find the null monic polynomials of degree 3

$$\begin{aligned} P_1(x) &= x(x-1)(x-\theta) = x^3 - (1+\theta)x^2 + \theta x \\ P_2(x) &= x^2(x-1) = x^3 - x^2. \end{aligned}$$

Their difference

$$P_2(x) - P_1(x) = \theta x^2 - \theta x = \theta x(x-1) \tag{4.7}$$

is a null polynomial of M of degree 2, but it is not a monic polynomial.

Moreover, in this example we have an application of theorem 3.1. The minimum polynomial of the numeric part $\bar{M}$ is $m_{\bar{M}}(x) = x(x-1)$ and thus the two null polynomials of degree 3 are related by

$$P_1(x) = P_2(x) - \theta m_{\bar{M}}(x)$$

where $\theta = \theta_1\theta_2 \in \Lambda_{\bar{0}}$.

Moreover, we can check directly that there does not exist a null monic polynomial of the supermatrix M of degree two, that is, any equation of the form

$$M^2 + uM + vI = 0 \qquad u, v \in \Lambda_{\bar{0}}$$

is impossible.

Therefore, we can consider as the minimum polynomial of the supermatrix M one of the polynomials

$$P_1(x) = x(x-1)(x-\theta) \qquad P_2(x) = x^2(x-1).$$

The polynomial $P_1(x)$ contains all linear factors of the characteristic polynomial $P(x)$, while the polynomial $P_2(x)$ does not contain all linear factors of $P(x)$. However, both polynomials $P_1(x)$ and $P_2(x)$ can be constructed as products of factors selected from the characteristic polynomial $P(x) = x^3(x-1)^3(x-\theta)^6$.

Example 3. We consider the supermatrix

$$M = \begin{pmatrix} 1 & 0 & 0 & \theta_1 \\ 0 & 0 & \theta_2 & 0 \\ 0 & \theta_1 & 0 & 0 \\ \theta_2 & 0 & 0 & 0 \end{pmatrix} \tag{4.8}$$

given in [1], for which with the method proposed in [1], we find first the null monic polynomial

$$Q_1(x) = x^6 - (1+2\theta)x^5 + \theta x^3.$$

However, as noted in [1], some accidental cancellations which occur in this case, that are not possible to describe in general, lead finally to a lower degree null monic polynomial

$$Q(x) = x^4 - (1+2\theta)x^3 + \theta x.$$

We note that in this example we have

$$\begin{aligned} a(x) &= x(x-1) & d(x) &= x^2 \\ \tilde{F}(x) &= x^3[x^2(x-1)-\theta] & \tilde{G}(x) &= x^6 \\ F(x) &= x^3(x-1)^3 & G(x) &= x^2(x-1)[x^2(x-1)+\theta]. \end{aligned}$$

Alternatively, we have the characteristic polynomial

$$P(x) = \tilde{F}(x)G(x) = x^3(x-1)[x^2(x-1)-\theta][x^2(x-1)+\theta] = x^9(x-1)^3$$

and using factors from this polynomial we find the null monic polynomial of the supermatrix M,

$$m(x) = x^4 - x^3 - \theta x = x[x^2(x-1)-\theta]$$

which is of the same degree as the polynomial $P(x)$. Moreover, it is a product of factors selected from the following factorization of the characteristic polynomial:

$$\begin{aligned} P(x) &= x^9(x-1)^3 = x^5x^4(x-1)^2(x-1) \\ &= x^5[x^2(x-1)-\theta][x^2(x-1)+\theta](x-1). \end{aligned}$$

The same is true for the polynomial $Q(x)$ according to the factorization

$$Q(x) = x^4 - (1+2\theta)x + \theta x = x[x^2(x-1) - \theta(2x^2-1)]$$

and

$$\begin{aligned} P(x) &= x^9(x-1)^3 = x^5(x-1)[x^2(x-1)]^2 \\ &= x^5(x-1)[x^2(x-1)-\theta(2x^2-1)][x^2(x-1)+\theta(2x^2-1)]. \end{aligned}$$

Moreover, we can write

$$Q(x) = m(x) - 2\theta x(x-1)(x+1).$$

The difference

$$m(x) - Q(x) = 2\theta x(x-1)(x+1)$$

is a polynomial annihilating M, but it is not a monic polynomial.

We consider the numeric part $\bar{M}$ of the supermatrix M, i.e. the matrix formed by the numeric parts of the entries of M. We have

$$\bar{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It has characteristic polynomial $\mathcal{X}_{\bar{M}}(x) = x^3(x-1)$ and minimum polynomial $m_{\bar{M}}(x) = x(x-1)$. Therefore we have the equation

$$m(x) = Q(x) + 2\theta(x+1)m_{\bar{M}}(x).$$

5. Summary

For an arbitrary supermatrix M there has been introduced by Urrutia and Morales [1] two types of null polynomials, thus providing an extension of the Cayley–Hamilton theorem for supermatrices.

The first one is the so-called characteristic polynomial $P(x)$, which is directly associated with the supermatrix M independent of the factorization of the numerator and denominator of the rational function $s\det(xI - M)$.

The second is the polynomial $Q(x)$ which has been introduced as the minimum polynomial of the supermatrix M. Its definition depends on the existence of a maximum common divisor of the polynomials $\tilde{F}(x)$ and $\tilde{G}(x)$ or $F(x)$ and $G(x)$ which are the building blocks of $s\det(xI - M)$. However, because of the elimination of the common factors R and S from the polynomials $\tilde{F}(x)$ and $\tilde{G}(x)$ or $F(x)$ and $G(x)$ we miss the possibility of choosing factors for the construction of a minimum polynomial of M from the complete set of divisors of the polynomials $\tilde{F}(x)$ and $\tilde{G}(x)$ or $F(x)$ and $G(x)$. It has been shown here that using factors from $\tilde{F}(x)$ and $\tilde{G}(x)$ or $F(x)$ and $G(x)$, it is possible to determine a null polynomial of M of a degree lower than the degree of the polynomial $Q(x)$.

The case of $(1+1) \times (1+1)$ supermatrices has been studied completely. The form of the minimum polynomial has been described in the general case of $(m+n) \times (m+n)$ supermatrices. Since the minimum polynomial divides the characteristic polynomial, this can be constructed as a product of factors arising from any possible factorization of the characteristic polynomial. The minimum polynomial does not necessarily contain all linear factors involved in a particular factorization of the characteristic polynomial (example 2), a result which is due to the nature of the elements of the off-diagonal blocks B and C, which are nilpotent of degree 2. In addition, as a consequence of theorem 3.1, we get that the null polynomial of minimum degree of a supermatrix is not necessarily unique.

References

[1] Urrutia L F and Morales N 1994 *J. Phys. A: Math. Gen.* **27** 1981
[2] Horne J H 1990 *Nucl. Phys.* B **334** 669
[3] Birmingham D, Blau M, Rakowski M and Thompson G 1991 *Phys. Rep.* **209** 129
[4] Witten E 1988 *Nucl. Phys.* B **331** 46
[5] Koehler K, Mansouri F, Vaz C and Witten L 1990 *Mod. Phys. Lett.* A **5** 935
Koehler K, Mansouri F, Vaz C and Witten L 1990 *Nucl. Phys.* B **341** 167
[6] Armand-Ugon D, Gambini R, Obregon O and Pullin J 1996 *Nucl. Phys.* B **460** 615
[7] Mostow G D, Sampson J H and Meyer J P 1963 *Fundamental Structures of Algebra* (New York: McGraw-Hill)
[8] Rogers A 1980 *J. Math. Phys.* **21** 1352

[9] Berezin F A 1987 *Introduction to Superanalysis* (Dordrecht: Reidel)
[10] van Nieuwenhuizen P 1981 *Phys. Rep.* 68**C** 189
[11] Backhouse N B and Fellouris A G 1984 *J. Phys. A: Math. Gen.* **17** 1389
[12] Arnowitt R, Nath P and Zumino B 1975 *Phys. Lett.* **56** 81
[13] De Witt B 1984 *Supermanifolds* (Cambridge: Cambridge University Press) p 33
[14] Kobayashi Y and Nagamachi S 1990 *J. Math. Phys.* **31** 2726
[15] Kobayashi Y and Nagamachi S 1988 *Proc. Japan Acad.* **64** 249
[16] Urrutia L F and Morales N 1994 *Lett. Math. Phys.* **32** 211
[17] Lang S 1965 *Algebra* (Reading, MA: Addison-Wesley)